Quantum Reflection of S-wave Unstable States

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Abstract. The phase time in quantum tunneling can be disentangled into a dwell time plus a term arising due to the interference of the reflected and incident waves in front of the barrier. The interference term dominates at low energies and as $E \to 0$, this term and hence the phase time becomes singular. With the *s*-wave motion in three dimensions being equivalent to that of a one-dimensional motion in the radial coordinate, a similar singularity shows up in the phase time delay of *s*-wave resonances. Relating the scattering matrix in three dimensional collisions to the reflection amplitude, the interference term in tunneling can be identified as a term given in terms of the transition matrix in scattering. Subtraction of this term from the phase time delay gives the dwell time delay which is finite at all energies and is useful in characterizing *s*-wave resonances such as the σ meson and mesic nuclei near threshold.

TUNNELING TIMES

Though the tunneling of a particle through a classically forbidden region is one of the oldest quantum phenomenon observed, the amount of time spent by a particle in the barrier region has remained a controversial topic over decades. The classical definition for the duration of a collision event for example is straightforward, however, the quantum mechanical one depends on the approach used. Hence, in an attempt to answer the question of how long does a particle need to traverse the barrier, several different tunneling times were defined [1]. One of the earliest definition was that of a *phase time* which involves following the peak of a narrow wave packet. An average *dwell time* in scattering collisions was introduced by Smith [2] in 1960 and was discussed later in the context of one dimensional (1D) tunneling problems by Büttiker [3]. The average was taken over reflection and transmission in 1D and over the scattering channels in three dimensions. In 1966 Baz' proposed a Gedanken experiment where a small magnetic field is confined to a region and the Larmor precession of electrons is used as a quantum clock. Rybachenko applied the method to discuss the *Larmor time* [4] in 1D. If the electrons have a direction of polarization perpendicular to the direction of the field, the time spent in the field region was found to be proportional to the expectation value of a spin component. Büttiker and Landauer defined a *traversal time* [5] which turns out to be formally similar to the inverse of the assault frequency in a tunneling problem [6]. Recently the tunneling time concepts have gained importance in the discussions of superluminal propagation of tunneling particles which by itself is a controversial topic [7]. The Hartmann effect connected with superluminality involves the saturation of dwell time with the barrier width. Since the time becomes independent of the barrier width, a thick enough barrier can lead to superluminal propagation. A review and some recent

applications of tunneling times can be found in [8].

Dwell and phase time connection

The dwell and asymptotic phase times are the most commonly applied time concepts which seem to provide reliable and complementary information on time aspects of tunneling processes. In a 1D treatment of tunneling through a barrier, following the peak of a sharp transmitted wave packet, $T(k)e^{i\phi_T(k)}e^{ikx-iE(k)t/\hbar}$, one finds [1] that the time difference between the arrival and departure of the wave packet at the barrier is given by the energy derivative of the transmission phase. A similar analysis for the reflected wave leads to a reflection phase time given by the energy derivative of the reflection phase. A weighted sum of the two possibilities (sometimes known as group delay [9]) is then given by $\tau_\phi = |T|^2\tau_{\phi_T} + |R|^2\tau_{\phi_R}$, where $|T|^2$ and $|R|^2$ are the transmission and reflection probabilities, ϕ_T and ϕ_R are the transmission and reflection phases, $\tau_{\phi_T} = d\phi_T/dE$ and $\tau_{\phi_R} = d\phi_R/dE$. The average dwell time in 1D which is defined as the number of particles divided by the incident flux can be shown to be related to the above phase times. Such a relation for a particle with an incident energy, $E = \hbar^2k^2/2\mu$, is easily obtained after some rearrangement of the Schrödinger equation and is given by [1, 9],

$$\tau_\phi(E) = \tau_D(E) - \hbar\,[Im(R)/k]\,dk/dE\,. \tag{1}$$

The average dwell time for the barrier region extending from x_1 to x_2 for example is given by $\tau_D(E) = \int_{x_1}^{x_2} |\Psi|^2/j$, where $j = \hbar k/\mu$. The second term on the right of (1) is the self-interference term which arises due to the overlap of the incident and reflected waves in front of the barrier. This term is important at low energies and becomes singular as $E \to 0$, thus making the phase time singular too. A nice demonstration of the above was done [1] for the case of an opaque rectangular barrier ($T \ll 1$), where it was shown that the phase time and dwell times are given as,

$$\tau_\phi \simeq \frac{2\mu}{\hbar\kappa k}\,,\ \tau_D \simeq \frac{2\mu k}{\hbar\kappa k_0^2},\ \text{with}\ \kappa^2 = k_0^2 - k^2 \tag{2}$$

so that $\tau_\phi \overset{k\to 0}{\simeq} \infty$ and $\tau_D \overset{k\to 0}{\simeq} 0$.

DELAY TIMES

The tunneling times considered in the previous section represent the times spent by the tunneling particle interacting with the barrier. If one subtracts the time spent by the particle in the absence of the barrier, one obtains a definition for time delay. Thus,

$$\tilde{\tau}_\phi(E) = \tilde{\tau}_D(E) - \hbar\,[Im(R)/k]\,dk/dE\,, \tag{3}$$

where, $\tilde{\tau}_\phi(E) = \tau_\phi(E) - \tau^0(E)$ and $\tilde{\tau}_D(E) = \tau_D(E) - \tau^0(E)$ are now the phase and the dwell time delay respectively. In the early fifties, Wigner defined [10] a time delay in

purely elastic scattering collisions. Following the peak of a scattered wave packet, this delay was given by the energy derivative of the scattering phase shift. Considering the analogy to the definitions in 1D tunneling, we shall refer to it henceforth as a "phase time delay" in three dimensions (3D). In 3D collisions, one finds a straightforward application of the time delay concept for resonance physics [11]. For example, if the elastic scattering of two particles a and b can proceed through the formation of an on-shell intermediate state R, which is formed at a time t_1 and decays ($R \rightarrow a + b$) at time t_2, then the process $a + b \rightarrow R \rightarrow a + b$ is "delayed" as compared to the non-resonant scattering process, $a + b \rightarrow a + b$, by an amount $\Delta t = t_2 - t_1$ which corresponds to the lifetime of the state R. In the presence of inelasticities, a one to one correspondence between time delay and the lifetime of a resonance does not exist and one rather defines a time delay matrix [2].

Density of states and dwell and phase times

The physical significance of the tunneling and collision times can be further understood through their connections to the density of states (DOS). A relation between the dwell time and the DOS for a 3D system of arbitrary shape with an arbitrary number of incoming channels was derived in [12]. The DOS, $\rho_{\Omega}^{3D}(E)$ in Ω, proportional to the sum of the dwell times in Ω for all incoming channels was shown to be,

$$\rho_{\Omega}^{3D}(E) = \frac{1}{2\pi\hbar} \sum_{n=1}^{N} \tau_D^n(E).$$ (4)

For the relation in 1D, the number of channels reduces to two and for a symmetric barrier, $\rho_{\Omega}^{1D}(E) = (1/\pi\hbar)\tau_D(E)$ [13].

The phase time delay or rather the scattering phase shift derivative in Wigner's time delay, namely, $\tilde{\tau}_\phi(E) = 2\hbar d\delta/dE$, is related through the Beth-Uhlenbeck formula to the density of states [14] as,

$$\rho_l(E) - \rho_l^0(E) = \left(\frac{2l+1}{\pi}\right)\frac{d\delta_l(E)}{dE},$$ (5)

where $\rho_l(E)$ and $\rho_l^0(E)$ are the densities of states with and without interaction respectively. The density of states with interaction can sometimes be less than that without interaction. This could happen for repulsive potentials or due to occurrence of large inelasticities in elastic collisions [15]. In such cases, the right hand side of the above equation would be negative and hence one would obtain a negative time delay. There exists however a lower bound on the negativity due to causality as shown by Wigner [10]. The interpretation of the phase time delay as a difference in the density of states however becomes problematic for the particular case of $l = 0$ as $E \rightarrow 0$. Since the scattering phase shift, $\delta \propto k^{2l+1}$, or rather $\delta \propto E^{l+1/2}$, the energy derivative $d\delta/dE \propto E^{l-1/2}$. Hence, for all $l \neq 0$, the phase time delay smoothly vanishes as $E \rightarrow 0$, however, for $l = 0$, $d\delta/dE \rightarrow \infty$ as $E \rightarrow 0$. This singularity is the same as the one which we already discussed in the previous section in connection with the phase time in 1D tunneling at

low energies. In the tunneling time relation, one can subtract the interference term in (3) from the phase time delay to obtain the dwell time delay which remains finite at all energies. Can one find an interference-like term in collisions to subtract from the phase time delay? To answer this question, let us first have a look at the lifetime matrix as defined by Smith.

Multichannel delay time matrix

Beginning with the quantum mechanical definition of the average time of residence, namely, the dwell time in a region, Smith defined the lifetime as the difference between these residence times with and without interaction. In an elegant derivation [2], he found a connection between the scattering matrix and a lifetime matrix, in 1D and 3D and for elastic as well as inelastic collisions. The lifetime matrix $\mathbf{Q}$ is related to $\mathbf{S}$ as, $\mathbf{Q} = i\hbar\mathbf{S}d\mathbf{S}^\dagger/dE = -i\hbar(d\mathbf{S}/dE)\mathbf{S}^\dagger = \mathbf{Q}^\dagger$, such that $\mathbf{Q}$ is Hermitian. Identifying $\mathbf{t} = -i\hbar\partial/\partial E$ as a time operator, he found, $\mathbf{Q} = -\mathbf{S}t\mathbf{S}^\dagger = (\mathbf{t}\mathbf{S})\mathbf{S}^\dagger$. The average delay experienced by a particle injected in the ith channel was given as, $\sum_j S_{ij}^* S_{ij}\Delta t_{ij} = \Re e\left(-i\hbar \sum_j S_{ij}^* dS_{ij}/dE\right) = Q_{ii}$, such that an element of the matrix $\Delta \mathbf{t}$ was given by, $\Delta t_{ij} = \Re e(-i\hbar(S_{ij})^{-1}dS_{ij}/dE)$, where S_{ij} is an element of the corresponding S-matrix. If we consider a 2×2 S-matrix, such that $S_{ii} = \eta\, e^{2i\delta_i}$ where δ_i is the real scattering phase shift for the elastic scattering in channel i and η is the inelasticity parameter (with $0 < \eta \leq 1$), then the diagonal elements of the time delay matrix are $\Delta t_{ii} = d\delta_i/dE$ and *have the meaning of a "phase time delay" in a given channel*. Re-writing the $\mathbf{S}$ matrix as, $\mathbf{S} = 1 + 2i\mathbf{T}$, one can also find an expression for $\Delta \mathbf{t}$ in terms of $\mathbf{T}$. With $T = -(\mu k/2\pi)t$, for the time delay in s-wave elastic collisions, a single element, $\Delta t_{ii}^{l=0}$ ($= \tilde{\tau}_\phi(E)$) is given as,

$$\tilde{\tau}_\phi(E) = \frac{2\hbar}{A}\left[\frac{-\mu}{2\pi}k\frac{dt_R}{dE} - \frac{\mu^2 k^2}{2\pi^2}\left(t_I\frac{dt_R}{dE} - t_R\frac{dt_I}{dE}\right) - \frac{\mu}{2\pi}t_R\frac{dk}{dE}\right], \tag{6}$$

with $A = 1 + (2\mu k t_I/\pi) + (\mu^2 k^2(t_R^2 + t_I^2)/\pi^2)$. For elastic scattering in the absence of inelasticities, the factor $A = 1$. In the above equation, $\tilde{\tau}_\phi(E)$ can once again be seen to blow up as $E \to 0$.

Before going over to the next section, we note that the S_{ij}'s in Smith's expression are in general elements of a multichannel S-matrix, for a given partial wave l. If one does not perform a partial wave expansion, one obtains an energy and angle dependent time delay of the full wave packet. A detailed analysis of various multichannel time delay concepts can be found in [16].

THE SINGULARITY AND QUANTUM REFLECTION

The phase time delay which becomes singular near threshold in the case of s-wave elastic scattering poses a problem for the resonances occurring near threshold and the interpretation as a density of states is no more useful. In order to resolve the problem

and subtract the singularity as in the case of 1D tunneling, we first notice the following: with there being no angle dependence of the scattering amplitude in the case of s-waves, the s-wave 3D motion can be viewed as a 1D motion in the radial coordinate r. Having translated the problem to a 1D one, we relate a 2-channel S-matrix to the reflection and transmission amplitudes $R(E)$ and $T(E)$ respectively. Considering an asymptotic wave function with incidence from the left (L),

$$
\begin{aligned}
\Psi_k(x) &= e^{ikx} + R_L(E)\,e^{-ikx} \quad x \to -\infty \\
&= T_L(E)\,e^{ikx} \quad x \to +\infty
\end{aligned}
\tag{7}
$$

and the S-matrix is given as,

$$
S = \begin{pmatrix} T_L(E) & -R_R(E) \\ -R_L(E) & T_R(E) \end{pmatrix}
\tag{8}
$$

where, $T_L(E) = T_R(E) = T(E)$ due to time reversal invariance and $R_L(E) = R_R(E) = R(E)$ for symmetric potentials. Substituting for this S-matrix in Smith's time delay relation, we obtain, $Q_{11} = |T|^2 \Delta t_{11} + |R|^2 \Delta t_{12}$ and $Q_{22} = |R|^2 \Delta t_{21} + |T|^2 \Delta t_{22}$.

Having defined the time delay matrix as above, we now resort to the concept of quantum reflection which corresponds to the reflection of a particle in a classically allowed region where there is no classical turning point. If we associate the amplitude $R(E)$ with such a reflection, we can indeed use the above time delay matrix for an s-wave resonance in the absence of a potential barrier too. Quantum reflection dominates at low energies and in *badlands* where the semi-classical condition

$$
\Delta(x) = \frac{1}{2\pi}\left|\frac{d\lambda}{dx}\right| = \hbar\left|\frac{\mu}{k^3}\frac{dV}{dx}\right| << 1
\tag{9}
$$

is no more valid. Here λ is the de Broglie wave length and k the wave number. Since the transmission coefficient becomes negligible at low energies, to a good approximation one can assume that $S \to -R$ [17] and with, $S = 1 - i\mu k\,(t_R + it_I)/\pi$, where t_R and t_I are the real and imaginary parts of the t-matrix respectively and μ is the reduced mass of the system, we obtain the 'self-interference' term of the 1D tunneling problem in terms of t as [17],

$$
-\hbar\frac{Im(R)}{k}\frac{dk}{dE} = -\hbar\mu\frac{t_R}{\pi}\frac{dk}{dE}.
\tag{10}
$$

One can thus identify a dwell time delay in s-wave elastic scattering in the absence of inelasticities as, $\tilde{\tau}_D(E) = \tilde{\tau}_\phi(E) + \hbar\mu\,[t_R/\pi]\,dk/dE$. $\tilde{\tau}_D(E)$ is finite at all energies and as seen before [12] also has the interpretation of the difference of DOS with and without interaction. Since the reflection related term dominates at low energies, the phase and dwell time delay become equal at high energies.

Applications

An application of this result in the search for eta-mesic states of nuclei was demonstrated in [17]. The removal of the singularity from the phase time delay helped in char-
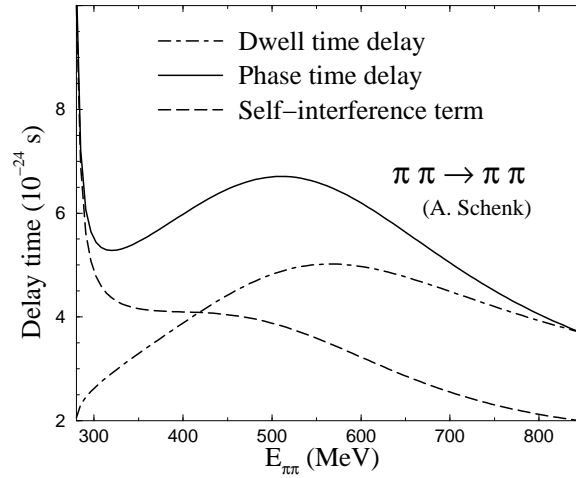
FIGURE 1. Delay times in $\pi\pi$ elastic scattering producing the near threshold σ resonance.

acterizing the near threshold states of eta mesons and helium nuclei. Here, we show the time delay in $\pi\pi$ elastic scattering which is evaluated using the parameterization of the $\pi\pi$ elastic scattering t-matrix as given in [18]. The interpretation of the delay times in terms of density of states becomes important for a semi-empirical determination of the non-exponential decay law at large times [19]. The case of threshold resonances which are very broad is particularly interesting [20] and dwell time delay can be used as an input in the evaluation of the survival probability of the threshold resonances. As seen in Fig. 1, at large energies the two delay time definitions are the same.

REFERENCES

1. E. H. Hauge and J. A. Støvneng, *Rev. Mod. Phys.* **61**, 917–936 (1989).
2. F. T. Smith, *Phys. Rev.* **118**, 349–356 (1960).
3. M. Büttiker, *Phys. Rev.* **B 27**, 6178–6188 (1983).
4. A. I. Baz, *Yad. Fiz.* **5** 229–235 (1967); V. F. Rybachenko, Yad. Fiz. **5**, 895–901 (1967).
5. M. Büttiker and R. Landauer, Phys. Rev. Lett. **49**, 1739–1742 (1982).
6. N. G. Kelkar and H. M. Castañeda, Phys. Rev. **C 76**, 064605–8 (2007).
7. H. G. Winful, *Phys. Rep.* **436**, 1–69 (2006).
8. V. S. Olkhovsky and E. Recami, *Phys. Rep.* **214**, 339–356 (1992); W. O. Amrein and Ph. Jacquet, *Phys. Rev.* **A 75**, 022106-20 (2007); W. Bauer and G. F. Bertsch, *Phys. Rev. Lett.*; **65**, 2213–2216 (1990); C. A. A. de Carvalho and H.M. Nussenzveig, *Phys. Rep.* **364**, 83–174 (2002).
9. H. G. Winful, *Phys. Rev. Lett.* **91**, 260401–4 (2003).
10. E. P Wigner, *Phys. Rev.* **98**, 145–147 (1955).
11. N. G. Kelkar, M. Nowakowski, K. P. Khemchandani and S. R. Jain, *Nucl. Phys.* **A 730**, 121–140 (2004); N. G. Kelkar, M. Nowakowski and K. P. Khemchandani, *Mod. Phys. Lett.* **A 19**, 2001–2008 (2004); *ibid*, *Nucl. Phys.* **A 724**, 357–374 (2003); *ibid*, *J. Phys.* **G 29**, 1001–1009 (2003).
12. G. Iannaccone, *Phys. Rev.* **B 51**, R4727–R4729 (1995).
13. V. Gasparian and M. Pollak, *Phys. Rev.* **B 47**, 2038–2041 (1993).
14. E. Beth and G. E. Uhlenbeck, Physics **4**, 915 (1937); K. Huang, *Statistical Mechanics*, Wiley, New York (1963).
15. N. G. Kelkar, *J. Phys.* **G29** L1–L8 (2003); N. G. Kelkar, K. P. Khemchandani and B. K. Jain, *J. Phys.* **G 32**, 1157–1170 (2006).

16. N. G. Kelkar and M. Nowakowski, Phys. Rev. **A 78**, 012709 (2008); arXiv: 0805.0608.
17. N. G. Kelkar, *Phys. Rev. Lett.* **99** 210403–4 (2007).
18. A. Schenk, *Nucl. Phys.* **B 363** , 97–113 (1991).
19. N. G. Kelkar, M. Nowakowski and K. P. Khemchandani *Phys. Rev.* **C 70**, 024601–4 (2004).
20. M. Nowakowski and N. G. Kelkar, AIP Conf. Proc. 1030, 250 (2008); arxiv: 0807.5103.